B.Sc. (Math) part II
paper – III

Topic: – Kernal of the homomorphism

Def: – Let $f: G \to G_1$ be a homomorphism of groups. The subset H of G consisting of all those elements which are mapped into the identity of the group G_1 is called the Kernal of homomorphism f and is denoted by Ker f.

i.e $K = \text{Ker} f = \{x \in G \mid f(x) = e'\}$ where e' is identity in G_1.

Theorem Let $f: G \to G'$ be homomorphism of groups. Let K be the Kernal of f. Then K is a normal subgroup of G.

Proof: – Let f be a homomorphism of a group G into G'. Let e, e' be the identities of G and G' respectively. Let K be the Kernal of f. Then $K = \{x \in G : f(x) = e'\}$

First we shall prove that

K is subgroup of G

Let $a, b \in K$ then $f(a) = e'$
and $f(b) = e'$

Since K is Kernal of f
we have $f(ab^{-1}) = f(a)f(b^{-1})$
since f is a homomorphism

$= f(a)[f(b)]^{-1}$

$= e'(e')^{-1} = e'.e' = e'$

$\therefore ab^{-1} \in K$

Thus $a, b \in K \Rightarrow ab^{-1} \in K$. Hence K is subgroup of G. Now we want to show that K is a normal subgroup of G. Let g be any element of G and k be any element of K.

Then $f(k) = e'$, since K is the Kernal of f

we have $f(gkg^{-1}) = f(g)f(k)f(g^{-1})$
since f is a homomorphism

$= f(g)e'f(g^{-1})$

$= f(g)f(g^{-1}) = f(gg^{-1})$

$= f(e) = e$

which means that $gkg^{-1} \in K$
Hence K is normal subgroup of G.

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Theorem ② The necessary and sufficient condition for a homomorphism f of a group G into group a group H with Kernel K to be an isomorphism of G into H is that $K = \{e\}$

proof :- Let f be a homomorphism of group G into a group H. let e, e' be the identities of G and H respec-
-tively

let K be the kernal of f i.e
$K = \{x \in G : f(x) = e'\}$

Suppose f is isomorphism of G into H then f is one-one
let $a \in K$ then $f(a) = e'$ by def of Kernal
$\Rightarrow f(a) = f(e) \quad \because f(e) = e'$
$\Rightarrow a = e \quad f$ is one-one
Thus $a \in K \Rightarrow a = e$ This means that e is the only element of G which belongs to K. Therefore $K = \{e\}$

Conversely suppose that $K = \{e\}$. In order to prove that f is an iso-morphism of G into H. we need to prove that f is one-one.

let $a, b \in G$ then
$f(a) = f(b) \Rightarrow f(a)[f(b^{-1})]$
$= f(b)[f(b^{-1})]$
$= f(a) f(b^{-1}) = e'$ f is homomorphism

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$ by def of Ker

$\Rightarrow ab^{-1} = e \quad \because K = \{e\}$

$\Rightarrow ab^{-1}b = eb \Rightarrow a = b$

Thus $f(a) = f(b) \Rightarrow a = b$ and

f is one-one

Hence f is an isomorphism

G into H.